L19 Feb 26 Cpt

Wednesday, February 25, 2015 4:48 PM

Qu. How is $[a,b]$ different from (a,b) or $[a,\infty)$?

closed & bdd — open — unbounded

For the concept of bounded, need metric

i.e., $\exists\ x \in X,\ R > 0$ s.t. $A \subset B(x,R)$

Clearly, expect to remove the metric

Qu. What is the concept?

A. Heine-Borel

B. Bolzano-Weierstrass

C. Sequentially Compact

Given a topological space $(X, \mathcal{T})$.

A set $\mathcal{G} \subset \mathcal{T}$ is an open cover if

$X = \bigcup \mathcal{G}$ ← the union of all open sets in $\mathcal{G}$

A subset $\mathcal{E} \subset \mathcal{G}$ is a subcover if

it is already an open cover, i.e., $\bigcup \mathcal{E} = X$

Heine-Borel

The space $(X, \mathcal{T})$ is compact if every open cover has a finite subcover

$\forall\ \mathcal{G} \subset \mathcal{T}$ with $\bigcup \mathcal{G} = X$, $\exists$ finite $\mathcal{E} \subset \mathcal{G}$ such that $\bigcup \mathcal{E} = X$.

Let us also recall the other two concepts.

Bolzano-Weierstrass

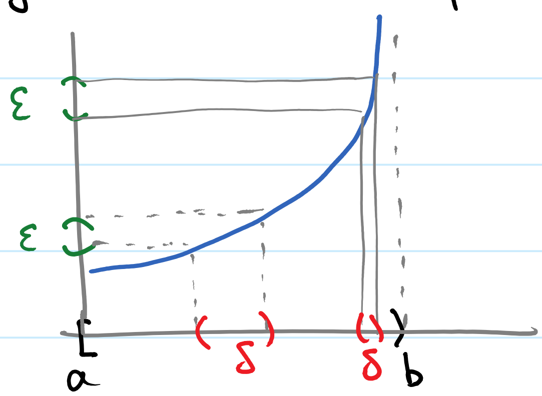
Every infinite set in X has a cluster point.

Sequentially compact

Every sequence has a convergent subsequence.

Each of the three concepts has its importance and usefulness. Let us use the following example to understand **Heine-Borel**.

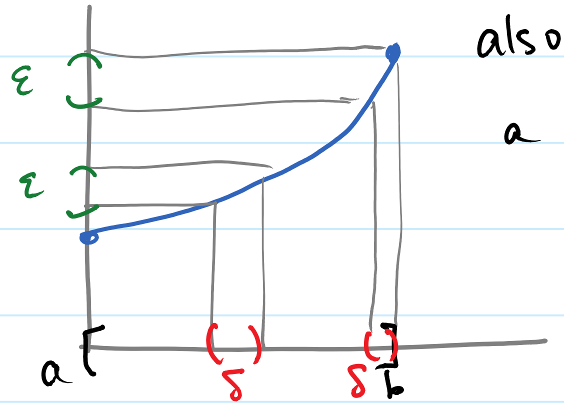
Continuity — The $\delta > 0$ depends on $\varepsilon > 0$ and x



For the same $\varepsilon > 0$, $\delta > 0$ gets smaller and smaller.

There are infinitely many δ-intervals on $[a, b)$

For uniformly continuity, though the δ-intervals also get smaller. There is a minimum size. Therefore $[a, b]$ can be covered by finitely many δ-intervals



Examples

1. $[a,b]$ is compact. How to prove it?
2. $\mathbb{R}^n$ is not compact
   - $\mathbb{R}^n$ always has a finite open cover, $\mathcal{G}=\{\mathbb{R}^n\}$
     This is irrelevant to compactness
   - $\mathbb{R}^1 = \bigcup_{n\in\mathbb{Z}} (2n, 2n+2) \cup (2n+1, 2n+3)$ but cannot be reduced to finitely many.
   - $\mathcal{G} = \{ B(m,1) : m \in \mathbb{Z}\times\mathbb{Z} \}$ is an open cover for $\mathbb{R}^2$
     Can we take away some sets from $\mathcal{G}$?
3. $(0,1] = \bigcup_{n\in\mathbb{N}} (\frac{1}{n}, 1]$ is not compact
4. Qu. Is this compact?
   $$K = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$$
   In general, let $x_n \longrightarrow x$ in X. Then
   $$K = \{x\} \cup \{x_n : n \in \mathbb{N}\} \text{ is compact}$$

Compact Subset

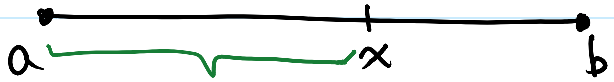
Given $(X, \mathcal{T})$ and $A \subset X$.

$(A, \mathcal{T}|_A)$ is compact $\Longleftrightarrow$

$\forall\, \mathcal{G} \subset \mathcal{T}$ with $\bigcup \mathcal{G} \supset A$, $\exists$ finite $\mathcal{E} \subset \mathcal{G}$ such that $\bigcup \mathcal{E} \supset A$

Recall $[a,b]$ is compact

Let $\mathcal{G} \subset \mathcal{J}$ be an open cover for $[a,b]$

① 

Ask if $[a,x]$ can have finite subcover

$$T = \left\{ x \in [a,b] : [a,x] \text{ can be covered by finite } \mathcal{F} \subset \mathcal{G} \right\}$$

Then $T \neq \emptyset$ because $a \in T$

Let $s = \sup T$, exists and $s \leq b$

Can prove that $s < b$ gives contradiction

② $[a,b] \to \left[a, \frac{a+b}{2}\right]$, $\left[\frac{a+b}{2}, b\right]$ — One of them cannot have finite subcover

Get $[a,b] \supset [a_1, b_1] \supset \cdots \supset [a_k, b_k] \supset \cdots$

If it stops at finite step then done

If it does not stop then contradiction

Note. Second method is valid for closed & bdd subset in $\mathbb{R}^n$, or totally bounded complete metric space

Qu. Observe from examples in $\mathbb{R}$, is there any relation between closed & compact?

Theorem. If $(X,\mathcal{T})$ is compact and $A\subset X$ is closed then A is compact

Proof. Let $\mathcal{G}\subset X$ with $\bigcup\mathcal{G}\supset A$

$\vdots$ $\leftarrow$ $\mathcal{G}'=\mathcal{G}\cup\{X\setminus A\}$, $\bigcup\mathcal{G}'=X$

Get finite $\mathcal{E}\subset\mathcal{G}$

$\vdots$

$\bigcup\mathcal{E}\supset A$

Theorem If $f:(X,\mathcal{T}_X)\longrightarrow Y$ is continuous and X is compact then so is $f(X)\subset Y$.

Proof Let $\mathcal{G}\subset\mathcal{T}_Y$ with $\bigcup\mathcal{G}\supset f(X)$.

Then $\mathcal{G}_X=\{f^{-1}V : V\in\mathcal{G}\}$, $\bigcup\mathcal{G}_X=X$

$\therefore\ \exists\ \{f^{-1}V_1,\cdots,f^{-1}V_n\}\subset\mathcal{G}_X$ satisfies

$$\bigcup_{k=1}^{n} f^{-1}V_k=X$$

Set Theory

$$\bigcup_{k=1}^{n} V_k\supset f(X)$$

* X compact, $X\xrightarrow{q}X/\sim\ \Rightarrow X/\sim$ is so.
* $\prod X_\alpha$ compact $\Longrightarrow$ each factor X_β is so.

Qu. What about the converses?

For quotient, e.g., $\mathbb{R}\longrightarrow S^1$.